

# Exact solution of the alternating XXZ spin chain with generic non-diagonal boundaries

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## Abstract

The integrable XXZ alternating spin chain with generic non-diagonal boundary terms specified by the most general non-diagonal  $K$ -matrices is studied via the off-diagonal Bethe Ansatz method. Based on the intrinsic properties of the fused  $R$ -matrices and  $K$ -matrices, we obtain certain closed operator identities and conditions, which allow us to construct an inhomogeneous  $T - Q$  relation and the associated Bethe Ansatz equations accounting for the eigenvalues of the transfer matrix.

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# 1 Introduction

There has been significant focus of efforts on solving integrable quantum spin chains for many years due to their numerous and still growing applications in string and super-symmetric Yang-Mills theories [1, 2, 3, 4], statistical physics [5, 6], low-dimensional condensed matter physics [7], and even some mathematical areas such as quantum groups [8]. Among them, the XXZ spin chain (with various spins) plays a fundamental or guiding role [9]. The Bethe Ansatz solution [10] of the spin- $\frac{1}{2}$  XXZ chain with periodic boundary condition (or closed chain) was first given by Orbach [11] and revisited by Yang and Yang [12] and many others. The  $s = 1$  integrable spin chain was first proposed by Zamalodchikov and Fateev [13]. Its generalization to arbitrary  $s$  cases was subsequently constructed via the fusion techniques [14, 15] based on the fundamental  $s = \frac{1}{2}$  representations of the Yang-Baxter equation [16, 5], an important equation which eventually led to the discovery of the Quantum Inverse Scattering Method [17] and Quantum Groups [8, 14, 15, 18, 19]. For the open spin- $\frac{1}{2}$  chain [20, 21], Sklyanin [22] proposed a systematic method to construct and to diagonalize a commuting transfer matrix, based on solutions of the boundary Yang-Baxter equation or the reflection equation (RE) [23]. Since then it directly stimulated a great deal of studies on the exact solutions of the quantum integrable models with open boundaries. A striking feature of the reflection equation is that it allows non-diagonal solutions [24, 25], which breaks the usual  $U(1)$ -symmetry (i.e., the total spin is not anymore conserved) and leads to the corresponding eigenvalue problem quite frustrated. Many efforts had been made [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] to approach this nontrivial problem. However, in a long period of time, the Bethe Ansatz solutions could only be obtained for either constrained boundary parameters [26] or special crossing parameters [27] associated with spin- $\frac{1}{2}$  chains or with spin- $s$  chains [40, 41, 42, 43].

Recently, a method for solving the eigenvalue problem of integrable models with generic boundary conditions, i.e., the off-diagonal Bethe Ansatz (ODBA) method was proposed in [44] ([45] for the details) and then several long-standing models [44, 46, 47] were solved. Subsequently, its applications to integrable models beyond  $A$ -type [48] and to the high spin XXX open chain [49] were performed, and the nested-version of ODBA for the models associated with  $su(n)$  algebra [50] was developed. In addition, the method for thermodynamic analysis based on the ODBA solutions [51] was also proposed. It should be noted that two

other promising methods, namely, the  $q$ -Onsager algebra method [35] and the separation of variables method [52] were also used to approach the spin- $\frac{1}{2}$  chains with generic integrable boundaries. Very recently based on the inhomogeneous  $T - Q$  relations [44] obtained by ODBA, the corresponding Bethe states, which have well-defined homogeneous limits, have been proposed for the open chains by the modified algebraic Bethe ansatz [53, 54] and by ODBA [55, 56].

The high spin chains with periodic and diagonal boundaries have been extensively studied in the literature [13, 14, 57, 58, 59]. So far the Bethe Ansatz solutions of the models with non-diagonal boundaries were known only for some special cases such as the boundary parameters obeying some constraint [40] or the crossing parameter (or anisotropy constant)  $\eta$  taking some special value (e.g., roots of unity) [41, 42, 49]. Moreover, the XXZ chain can be generalized to integrable alternating spin chain [59, 60, 61, 62, 63], i.e., an inhomogeneous chain with spin  $s$  at odd sites and spin  $s'$  at even sites. The alternating spin chains have many applications in lower dimensional quantum field theories such as the  $SU(2)$  principle Chiral model [64, 65] and the super-symmetric sine-Gordon model [66, 67]. In this paper we shall investigate the Bethe Ansatz solution of the integrable XXZ alternating spin chain with an arbitrary  $\eta$  and generic non-diagonal boundary terms specified by the most general non-diagonal  $K$ -matrices via the ODBA.

The outline of the paper is as follows. Section 2 serves as an introduction to our notations and some basic ingredients. After briefly reviewing the fusion procedures for the  $R$ -matrix [14, 19, 68] and the associated  $K$ -matrices from the fundamental spin- $\frac{1}{2}$  ones [57, 69, 70], we introduce the corresponding transfer matrix of the open XXZ alternating spin chain with the most generic non-diagonal boundary  $K$ -matrices and the fused transfer matrices following the method in [59]. In Section 3, using the method in [40] we derive the corresponding fusion hierarchy of the high spin fused transfer matrices and give certain closed operator product identities for the fundamental transfer matrix by using some intrinsic properties of the high spin  $R$ -matrix ( $R^{(l,l)}(u)$ ) and  $K$ -matrices ( $K^{\pm(\frac{1}{2})}(u)$ ). The asymptotic behavior of the transfer matrix is also obtained. The resulting conditions completely characterize the eigenvalues of the fundamental transfer matrix (as a consequence, also determine the eigenvalues of all the high spin fused transfer matrices). Section 4 is devoted to the construction of the inhomogeneous  $T - Q$  relation and the corresponding Bethe Ansatz equations (BAEs). In section 5, we summarize our results and give some discussions. Some detailed technical

proofs are given in Appendix A&B.

## 2 Transfer matrices

### 2.1 Fusion of the $R$ -matrices and the $K$ -matrices

Let us fixed a generic complex  $\eta$  and two positive numbers  $s, s' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ . Throughout,  $V_i$  denotes a  $(2l_i + 1)$ -dimensional linear space  $(\mathbb{C}^{2l_i+1})$  which endows an irreducible representation of the quantum algebra  $U_q(sl_2)$  [8] with spin  $l_i$ , where  $q = e^\eta$ . The definition of  $U_q(sl_2)$  and its spin- $l$  representation are given in Appendix A. The  $R$ -matrix  $R_{ij}^{(l_i, l_j)}(u)$ , denoted as the spin- $(l_i, l_j)$   $R$ -matrix, is a linear operator acting in  $V_i \otimes V_j$ . The  $R$ -matrix satisfies the following quantum Yang-Baxter equation (QYBE) [16, 5]

$$R_{12}^{(l_1, l_2)}(u - v) R_{13}^{(l_1, l_3)}(u) R_{23}^{(l_2, l_3)}(v) = R_{23}^{(l_2, l_3)}(v) R_{13}^{(l_1, l_3)}(u) R_{12}^{(l_1, l_2)}(u - v). \quad (2.1)$$

Here and below we adopt the standard notations: for any matrix  $A \in \text{End}(V)$ ,  $A_j$  is an embedding operator in the tensor space  $V \otimes V \otimes \dots$ , which acts as  $A$  on the  $j$ -th space and as identity on the other factor spaces;  $R_{ij}(u)$  is an embedding operator of  $R$ -matrix in the tensor space, which acts as identity on the factor spaces except for the  $i$ -th and  $j$ -th ones.

The fundamental spin- $(\frac{1}{2}, l)$   $R$ -matrix  $R_{12}^{(\frac{1}{2}, l)}(u)$  (also called the  $L$ -operator [9]) defined in spin- $\frac{1}{2}$  (i.e., two-dimensional) auxiliary space and spin- $l$  (i.e.,  $(2l + 1)$ -dimensional) quantum space is given by [14, 15]

$$\begin{aligned} R_{12}^{(\frac{1}{2}, l)}(u) &= \sinh \left( u + \eta \left( \frac{1}{2} + \sigma_1^3 S_2^3 \right) \right) + \sinh \eta (\sigma_1^+ S_2^- + \sigma_1^- S_2^+) \\ &= \begin{pmatrix} \sinh(u + \frac{\eta}{2} + \eta S_2^3) & \sinh \eta S_2^- \\ \sinh \eta S_2^+ & \sinh(u + \frac{\eta}{2} - \eta S_2^3) \end{pmatrix}, \end{aligned} \quad (2.2)$$

where  $\eta$  is the so-called crossing parameter,  $\sigma^3, \sigma^\pm$  are the Pauli matrices and  $S^3, S^\pm$  are the spin- $l$  realizations given by (A.3)-(A.5) of the generators of the quantum algebra  $U_q(sl_2)$ . For the simplest case, i.e.,  $l = \frac{1}{2}$  the corresponding  $R$ -matrix reads

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \begin{pmatrix} \sinh(u + \eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u + \eta) \end{pmatrix}. \quad (2.3)$$

Besides the QYBE (2.1), the above well-known trigonometric six-vertex  $R$ -matrix also enjoys the following properties,

$$\text{Initial condition : } R_{12}^{(\frac{1}{2}, \frac{1}{2})}(0) = \sinh \eta P_{12}, \quad (2.4)$$

$$\text{Unitary relation : } R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) R_{21}^{(\frac{1}{2}, \frac{1}{2})}(-u) = -\xi(u) \text{id}, \quad (2.5)$$

$$\xi(u) = \sinh(u + \eta) \sinh(u - \eta), \quad (2.6)$$

$$\text{Crossing relation : } R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) = V_1 \{ R_{12}^{(\frac{1}{2}, \frac{1}{2})}(-u - \eta) \}^{t_1} V_1, \quad V = -i\sigma^y, \quad (2.7)$$

$$\text{PT-symmetry : } R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) = R_{21}^{(\frac{1}{2}, \frac{1}{2})}(u) = \{ R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) \}^{t_1 t_2}, \quad (2.8)$$

$$\text{Antisymmetric-fusion conditions : } R_{12}^{(\frac{1}{2}, \frac{1}{2})}(-\eta) \propto P_{12}^-. \quad (2.9)$$

Symmetric-fusion conditions :

$$R_{12}^{(\frac{1}{2}, \frac{1}{2})}(\eta) \propto \text{Diag}(\cosh \eta, 1, 1, \cosh \eta) P_{12}^+. \quad (2.10)$$

Here  $R_{21}^{(\frac{1}{2}, \frac{1}{2})}(u) = P_{12} R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) P_{12}$  with  $P_{12}$  being the permutation operator between the tensor product space of the spin- $\frac{1}{2}$  vector spaces;  $P_{12}^\pm = \frac{1}{2}(1 \pm P_{12})$ ; and  $t_i$  denotes transposition in the  $i$ -th space. The property (2.10) allows one to construct the spin- $(j, l)$   $R$ -matrix by using the symmetric fusion procedure [14, 15]

$$\begin{aligned} R_{\{1 \dots 2j\} \bar{1}}^{(j, l)}(u) &= B_{1 \dots 2j} A_{1 \dots 2j} P_{1 \dots 2j}^+ \prod_{k=1}^{2j} \left\{ R_{k, \bar{1}}^{(\frac{1}{2}, l)}(u + (k - j - \frac{1}{2})\eta) \right\} \\ &\times P_{1 \dots 2j}^+ A_{1 \dots 2j}^{-1} B_{1 \dots 2j}^{-1}, \end{aligned} \quad (2.11)$$

where  $P_{1 \dots 2j}^+$  is the symmetric projector given by

$$P_{1, \dots, 2j}^+ = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left( \sum_{l=1}^k P_{lk} \right), \quad (2.12)$$

and the  $u$ -independent matrices  $B_{1 \dots 2j}$  and  $A_{1 \dots 2j}$  are given in [27]. It is remarked that the  $R$ -matrices in the products (2.11) are ordered in the order of increasing  $k$  and that the fused  $R$ -matrices (2.2) and (2.11) satisfy the associated QYBE (2.1). Direct calculation shows that the spin- $(l, l)$   $R$ -matrix is given by (B.6) [19, 71]. In particular, the fused spin- $(l, l)$   $R$ -matrix satisfies the following important properties

$$\text{Initial condition : } R_{12}^{(l, l)}(0) \propto \mathbf{P}_{12}, \quad (2.13)$$

$$\text{Fusion condition : } R_{12}^{(l, l)}(-\eta) \propto \mathbf{P}_{12}^{(0)}(l), \quad (2.14)$$

where  $\mathbf{P}$  is the permutation operator between the tensor product space of the spin- $l$  vector spaces, and the projector  $\mathbf{P}^{(0)}(l)$  is related to the projector  $\bar{\mathbf{P}}^{(0)}(l)$  given by (B.4). Namely, the projector  $\mathbf{P}^{(0)}(l)$  is given by

$$\mathbf{P}^{(0)}(l) = |\Phi_0^{(l)}\rangle\langle\Phi_0^{(l)}|, \quad |\Phi_0^{(l)}\rangle = \frac{1}{\sqrt{2l+1}} \sum_{k=0}^{2l} (-1)^k |l-k\rangle \otimes |-l+k\rangle, \quad (2.15)$$

where  $\{|m\rangle|m=l, l-1, \dots, -l\rangle$  forms an orthonormal basis of the spin- $l$  space. The very properties (2.13) and (2.14) are the analogs of (2.4) and (2.9) for the higher spin case.

Having defined the fused- $R$  matrices, one can analogously construct the fused- $K$  matrices by using the methods developed in [57, 69, 70] as follows. The fused  $K^-$  matrices (e.g the spin- $j$   $K^-$  matrix) is given by

$$\begin{aligned} K_{\{a\}}^{-(j)}(u) &= B_{a_1 \dots a_{2j}} A_{a_1 \dots a_{2j}} P_{a_1 \dots a_{2j}}^+ \prod_{k=1}^{2j} \left\{ \left[ \prod_{l=1}^{k-1} R_{a_l a_k}^{(\frac{1}{2}, \frac{1}{2})}(2u + (k+l-2j-1)\eta) \right] \right. \\ &\quad \left. \times K_{a_k}^{-(\frac{1}{2})}(u + (k-j-\frac{1}{2})\eta) \right\} P_{a_1 \dots a_{2j}}^+ A_{a_1 \dots a_{2j}}^{-1} B_{a_1 \dots a_{2j}}^{-1}, \end{aligned} \quad (2.16)$$

where  $R^{(\frac{1}{2}, \frac{1}{2})}(u)$  is given by (2.3) and the  $u$ -independent matrices  $B_{a_1 \dots a_{2j}}$  and  $A_{a_1 \dots a_{2j}}$  are given in [27]. In this paper we consider the most general non-diagonal spin- $\frac{1}{2}$   $K^-$  matrix  $K^{-(\frac{1}{2})}(u)$  given by [24, 25]

$$\begin{aligned} K^{-(\frac{1}{2})}(u) &= \begin{pmatrix} K_{11}^-(u) & K_{12}^-(u) \\ K_{21}^-(u) & K_{22}^-(u) \end{pmatrix}, \\ K_{11}^-(u) &= 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)), \\ K_{22}^-(u) &= 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)), \\ K_{12}^-(u) &= e^{\theta_-} \sinh(2u), \quad K_{21}^-(u) = e^{-\theta_-} \sinh(2u), \end{aligned} \quad (2.17)$$

where  $\alpha_-$ ,  $\beta_-$ ,  $\theta_-$  are some boundary parameters. It is noted that the products of braces  $\{\dots\}$  in the above equation are ordered in the order of increasing  $k$ . The fused  $K_{\{a\}}^{-(j)}(u)$  matrices satisfy the following reflection equation [23, 40]

$$\begin{aligned} R_{\{a\}\{b\}}^{(j,s)}(u-v) K_{\{a\}}^{-(j)}(u) R_{\{b\}\{a\}}^{(s,j)}(u+v) K_{\{b\}}^{-(s)}(v) \\ = K_{\{b\}}^{-(s)}(v) R_{\{a\}\{b\}}^{(j,s)}(u+v) K_{\{a\}}^{-(j)}(u) R_{\{b\}\{a\}}^{(s,j)}(u-v). \end{aligned} \quad (2.18)$$

The fused dual reflection matrices  $K^{+(j)}$  [22] are given by

$$K_{\{a\}}^{+(j)}(u) = \frac{1}{f^{(j)}(u)} K_{\{a\}}^{-(j)}(-u - \eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)} , \quad (2.19)$$

where the normalization factor  $f^{(j)}(u)$  is,

$$f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^l [-\xi(2u + (l + k + 1 - 2j)\eta)],$$

with the function  $\xi(u)$  given by (2.6). Particularly, the fundamental one  $K^{+(\frac{1}{2})}(u)$  is

$$K^{+(\frac{1}{2})}(u) = K^{-(\frac{1}{2})}(-u - \eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)} , \quad (2.20)$$

where  $\alpha_+$ ,  $\beta_+$ ,  $\theta_+$  are some other boundary parameters.

## 2.2 Open alternating spin chains and its fused ones

Periodic alternating spin chains were first studied in [60, 61, 62, 63, 64] and then generalized to the open chain case [22, 72, 59]. Following [22], one can construct the associated transfer matrix for an alternating XXZ spin chain [73], namely, an inhomogeneous chain with spin  $s$  at odd sites and spin  $s'$  at even sites. Let us denote the transfer matrix  $t^{(j, (s, s'))}(u)$  whose auxiliary space is spin- $j$  ( $(2j + 1)$ - dimensional) and each of its  $2N$  quantum spaces with alternative spins, for any  $j, s, s' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ . The fused (or the spin- $(j, (s, s'))$ ) transfer matrix  $t^{(j, (s, s'))}(u)$  can be constructed by the fused  $R$ -matrices and  $K$ -matrices as follows [22, 40]

$$t^{(j, (s, s'))}(u) = \text{tr}_{\{a\}} K_{\{a\}}^{+(j)}(u) T_{\{a\}}^{(j, (s, s'))}(u) K_{\{a\}}^{-(j)}(u) \hat{T}_{\{a\}}^{(j, (s, s'))}(u) , \quad (2.21)$$

where  $T_{\{a\}}^{(j, (s, s'))}(u)$  and  $\hat{T}_{\{a\}}^{(j, (s, s'))}(u)$  are the fused one-row monodromy matrices given by

$$\begin{aligned} T_{\{a\}}^{(j, (s, s'))}(u) &= R_{\{a\}, 2N}^{(j, s')}(u - \theta_{2N}) R_{\{a\}, 2N-1}^{(j, s)}(u - \theta_{2N-1}) \dots \\ &\quad \times R_{\{a\}, 2}^{(j, s')}(u - \theta_2) R_{\{a\}, 1}^{(j, s)}(u - \theta_1) , \\ \hat{T}_{\{a\}}^{(j, (s, s'))}(u) &= R_{1, \{a\}}^{(s, j)}(u + \theta_1) R_{2, \{a\}}^{(s', j)}(u + \theta_2) \dots \\ &\quad \times R_{2N-1, \{a\}}^{(s, j)}(u + \theta_{2N-1}) R_{2N, \{a\}}^{(s', j)}(u + \theta_{2N}) . \end{aligned}$$

Here  $\{\theta_j | j = 1, \dots, 2N\}$  are arbitrary free complex parameters which are usually called the inhomogeneous parameters. The QYBE (2.1), the reflection equation (2.18) and its dual

one which can be deduced by the correspondence (2.19) between the  $K^-$ -matrices and  $K^+$ -matrices leads to [22] that these transfer matrices commute for different values of spectral parameter, any  $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$  and any  $s, s' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ ,

$$\left[ t^{(j, (s, s'))}(u), t^{(j', (s, s'))}(v) \right] = 0. \quad (2.22)$$

Therefore  $\{t^{(j, (s, s'))}\}$  serve as the generating functionals of the conserved quantities of the associated model, whose Hamiltonian can be given by some derivative of the logarithm of the associated transfer matrix  $t^{(j, (s, s'))}(u)$  at some special points [59, 71, 73], and thus ensure the integrability of the model.

### 3 Fusion hierarchy and the operator identities

#### 3.1 Operator identities

In the following part of the paper, let us denote the fused transfer matrices  $\{t^{(j, (s, s'))}(u)\}$  given by (2.21) by  $\{t^{(j)}(u)\}$  for simplicity. One may verify that these fused transfer matrices obey the following fusion hierarchy relation following the method in [57, 69, 40]

$$t^{(\frac{1}{2})}(u) t^{(j-\frac{1}{2})}(u - j\eta) = t^{(j)}(u - (j - \frac{1}{2})\eta) + \delta^{(s, s')}(u) t^{(j-1)}(u - (j + \frac{1}{2})\eta),$$

$$j = \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (3.1)$$

where we have used the convention  $t^{(0)} = \text{id}$ . The coefficient function  $\delta^{(s, s')}(u)$ , which is related to the quantum determinant, is given by

$$\begin{aligned} \delta^{(s, s')}(u) &= 2^4 \frac{\sinh(2u - 2\eta) \sinh(2u + 2\eta)}{\sinh(2u + \eta) \sinh(2u - \eta)} \sinh(u + \alpha_-) \sinh(u - \alpha_-) \cosh(u + \beta_-) \\ &\quad \times \cosh(u - \beta_-) \sinh(u + \alpha_+) \sinh(u - \alpha_+) \cosh(u + \beta_+) \cosh(u - \beta_+) \\ &\quad \times \prod_{l=1}^N \sinh(u - \theta_{2l-1} + (\frac{1}{2} + s)\eta) \sinh(u + \theta_{2l-1} + (\frac{1}{2} + s)\eta) \\ &\quad \times \prod_{l=1}^N \sinh(u - \theta_{2l} + (\frac{1}{2} + s')\eta) \sinh(u + \theta_{2l} + (\frac{1}{2} + s')\eta) \\ &\quad \times \prod_{l=1}^N \sinh(u - \theta_{2l-1} - (\frac{1}{2} + s)\eta) \sinh(u + \theta_{2l-1} - (\frac{1}{2} + s)\eta) \\ &\quad \times \prod_{l=1}^N \sinh(u - \theta_{2l} - (\frac{1}{2} + s')\eta) \sinh(u + \theta_{2l} - (\frac{1}{2} + s')\eta). \end{aligned} \quad (3.2)$$



Using the recursive relation (3.1), we can express the fused transfer matrix  $t^{(j)}(u)$  in terms of the fundamental one  $t^{(\frac{1}{2})}(u)$  with a  $2j$ -order operator product relation as follows:

$$\begin{aligned}
t^{(j)}(u) &= t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta) t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta - \eta) \dots t^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta) \\
&\quad - \delta^{(s,s')}(u + (j - \frac{1}{2})\eta) t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta - 2\eta) \dots t^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta) \\
&\quad - \delta^{(s,s')}(u + (j - \frac{1}{2})\eta - \eta) t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta) t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta - 3\eta) \\
&\quad \times \dots t^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta) \\
&\quad \vdots \\
&\quad - \delta^{(s,s')}(u - (j - \frac{1}{2})\eta + \eta) t^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta) \dots t^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta + 2\eta) \\
&\quad + \dots
\end{aligned} \tag{3.3}$$

For examples, the first three fused transfer matrices are given by

$$t^{(1)}(u) = t^{(\frac{1}{2})}(u + \frac{\eta}{2}) t^{(\frac{1}{2})}(u - \frac{\eta}{2}) - \delta^{(s,s')}(u + \frac{\eta}{2}), \tag{3.4}$$

$$\begin{aligned}
t^{(\frac{3}{2})}(u) &= t^{(\frac{1}{2})}(u + \eta) t^{(\frac{1}{2})}(u) t^{(\frac{1}{2})}(u - \eta) - \delta^{(s,s')}(u + \eta) t^{(\frac{1}{2})}(u - \eta) \\
&\quad - \delta^{(s,s')}(u) t^{(\frac{1}{2})}(u + \eta),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
t^{(2)}(u) &= t^{(\frac{1}{2})}(u + \frac{3\eta}{2}) t^{(\frac{1}{2})}(u + \frac{\eta}{2}) t^{(\frac{1}{2})}(u - \frac{\eta}{2}) t^{(\frac{1}{2})}(u - \frac{3\eta}{2}) \\
&\quad - \delta^{(s,s')}(u + \frac{3\eta}{2}) t^{(\frac{1}{2})}(u - \frac{\eta}{2}) t^{(\frac{1}{2})}(u - \frac{3\eta}{2}) \\
&\quad - \delta^{(s,s')}(u + \frac{\eta}{2}) t^{(\frac{1}{2})}(u + \frac{3\eta}{2}) t^{(\frac{1}{2})}(u - \frac{3\eta}{2}) \\
&\quad - \delta^{(s,s')}(u - \frac{\eta}{2}) t^{(\frac{1}{2})}(u + \frac{3\eta}{2}) t^{(\frac{1}{2})}(u + \frac{\eta}{2}) \\
&\quad + \delta^{(s,s')}(u + \frac{3\eta}{2}) \delta^{(s,s')}(u - \frac{\eta}{2}).
\end{aligned} \tag{3.6}$$

Keeping the very properties (2.13) and (2.14) in mind and following the method developed in [50, 48], we can derive the following relations with the help of QYBE (2.1),

$$T_1^{(s)}(\theta_{2j-1}) T_2^{(s)}(\theta_{2j-1} - \eta) = \mathbf{P}_{21}^{(0)}(s) T_1^{(s)}(\theta_{2j-1}) T_2^{(s)}(\theta_{2j-1} - \eta), \quad j = 1, \dots, N. \tag{3.7}$$

$$\hat{T}_1^{(s)}(-\theta_{2j-1}) \hat{T}_2^{(s)}(-\theta_{2j-1} - \eta) = \mathbf{P}_{21}^{(0)}(s) \hat{T}_1^{(s)}(-\theta_{2j-1}) \hat{T}_2^{(s)}(-\theta_{2j-1} - \eta), \quad j = 1, \dots, N, \tag{3.8}$$

where we use  $T^{(s)}(u)$  (or  $\hat{T}^{(s)}(u)$ ) to denote the one-row monodromy matrix  $T^{(s,(s,s'))}(u)$  (or  $\hat{T}^{(s,(s,s'))}(u)$ ). The above relations and the RE (2.18) enable us to derive the following relations of the double-row monodromy matrix  $\mathbb{T}^{(s)}(u) = T^{(s)}(u)K^{-s}(u)\hat{T}^{(s)}(u)$

$$\begin{aligned} & \mathbb{T}_1^{(s)}(\theta_{2j-1}) R_{2,1}^{(s,s)}(2\theta_{2j-1} - \eta) \mathbb{T}_2^{(s)}(\theta_{2j-1} - \eta) \\ &= \mathbf{P}_{21}^{(0)}(s) \mathbb{T}_1^{(s)}(\theta_{2j-1}) R_{2,1}^{(s,s)}(2\theta_{2j-1} - \eta) \mathbb{T}_2^{(s)}(\theta_{2j-1} - \eta), \quad j = 1, \dots, N. \end{aligned} \quad (3.9)$$

Then the above relation and the dual of RE (which can be derived by the RE (2.18) and the correspondence (2.19)) give rise to that the spin- $s$  transfer matrix satisfy the following operator identities,

$$t^{(s)}(\theta_{2j-1}) t^{(s)}(\theta_{2j-1} - \eta) = \Delta^{(s)}(u) \big|_{u=\theta_{2j-1}} \times \text{id}, \quad j = 1, \dots, N, \quad (3.10)$$

where the function  $\Delta^{(l)}(u)$  is

$$\Delta^{(l)}(u) = \prod_{k=0}^{2l-1} \delta^{(s,s')}(u - (l - \frac{1}{2})\eta + k\eta), \quad (3.11)$$

and the function  $\delta^{(s,s')}(u)$  is given by (3.2). Using the similar method, we can derive that the spin- $s'$  transfer matrix satisfy the following operator identities,

$$t^{(s')}(\theta_{2j}) t^{(s')}(\theta_{2j} - \eta) = \Delta^{(s')}(u) \big|_{u=\theta_{2j}} \times \text{id}, \quad j = 1, \dots, N, \quad (3.12)$$

where the function  $\Delta^{(s')}(u)$  is given by (3.11).

Now let us derive some properties of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$ . For this purpose we first list some properties of the fundamental spin- $(\frac{1}{2}, l)$   $R$ -matrix (2.2) which are some analogs of (2.6) and (2.7) for a generic  $l$ . With the help of the commutation relations (A.1) of  $U_q(sl_2)$  and (A.6), we can derive the following generalized unitary relation (c.f., (2.6))

$$R_{12}^{(\frac{1}{2}, l)}(u) R_{21}^{(l, \frac{1}{2})}(-u) = -\sinh(u + (\frac{1}{2} + l)\eta) \sinh(u - (\frac{1}{2} + l)\eta) \times \text{id}. \quad (3.13)$$

Direct calculation shows that the  $R$ -matrix  $R_{12}^{(\frac{1}{2}, l)}(u)$  also satisfies the following properties

$$R_{12}^{(\frac{1}{2}, l)}(u) = V_1 \{ R_{12}^{(\frac{1}{2}, l)}(-u - \eta) \}^{t_1} V_1, \quad V = -i\sigma^y, \quad (3.14)$$

$$R_{12}^{(\frac{1}{2}, l)}(u + i\pi) = -\sigma_1^z R_{12}^{(\frac{1}{2}, l)}(u) \sigma_1^z. \quad (3.15)$$

It is easy to check that the fundamental spin- $\frac{1}{2}$   $K$ -matrices  $K^{\pm(\frac{1}{2})}(u)$  given by (2.17) and (2.20) enjoy the following properties

$$K^{-(\frac{1}{2})}(0) = \frac{1}{2} \text{tr}(K^{-(\frac{1}{2})}(0)) \times \text{id}, \quad (3.16)$$

$$K^{-(\frac{1}{2})}\left(\frac{i\pi}{2}\right) = \frac{1}{2} \text{tr}\left(K^{-(\frac{1}{2})}\left(\frac{i\pi}{2}\right)\sigma^z\right) \times \sigma^z, \quad (3.17)$$

$$K^{\pm(\frac{1}{2})}(u + i\pi) = -\sigma^z K^{\pm(\frac{1}{2})}(u) \sigma^z. \quad (3.18)$$

The above relations, the explicit expressions of the spin- $(\frac{1}{2}, s)$  and spin- $(\frac{1}{2}, s')$   $R$ -matrices given by (2.2) and the spin- $\frac{1}{2}$   $K$ -matrices given by (2.17) and (2.20) imply that the transfer matrix  $t^{(\frac{1}{2})}(u)$  satisfies the following properties:

$$t^{(\frac{1}{2})}(u + i\pi) = t^{(\frac{1}{2})}(u), \quad (3.19)$$

$$t^{(\frac{1}{2})}(-u - \eta) = t^{(\frac{1}{2})}(u), \quad (3.20)$$

$$\begin{aligned} t^{(\frac{1}{2})}(0) &= -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \\ &\times \prod_{l=1}^N \sinh(\theta_{2l-1} + (\frac{1}{2} + s)\eta) \sinh(-\theta_{2l-1} + (\frac{1}{2} + s)\eta) \\ &\times \prod_{l=1}^N \sinh(\theta_{2l} + (\frac{1}{2} + s')\eta) \sinh(-\theta_{2l} + (\frac{1}{2} + s')\eta) \times \text{id}, \end{aligned} \quad (3.21)$$

$$\begin{aligned} t^{(\frac{1}{2})}\left(\frac{i\pi}{2}\right) &= -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \\ &\times \prod_{l=1}^N \sinh\left(\frac{i\pi}{2} + \theta_{2l-1} + (\frac{1}{2} + s)\eta\right) \sinh\left(\frac{i\pi}{2} + \theta_{2l-1} - (\frac{1}{2} + s)\eta\right) \\ &\times \prod_{l=1}^N \sinh\left(\frac{i\pi}{2} + \theta_{2l} + (\frac{1}{2} + s')\eta\right) \sinh\left(\frac{i\pi}{2} + \theta_{2l} - (\frac{1}{2} + s')\eta\right) \times \text{id}, \end{aligned} \quad (3.22)$$

$$t^{(\frac{1}{2})}(u) \big|_{u \rightarrow \infty} = -\frac{\cosh(\theta_- - \theta_+) e^{\pm[(4N+4)u + (2N+2)\eta]}}{2^{4N+1}} \times \text{id} + \dots \quad (3.23)$$

The analyticities of the spin- $(\frac{1}{2}, s)$  and spin- $(\frac{1}{2}, s')$   $R$ -matrices and spin- $\frac{1}{2}$   $K$ -matrices and the property (3.23) imply that the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$ , as a function of  $u$ , is a trigonometric polynomial of degree  $4N + 4$ . The fusion hierarchy relation (3.1) gives rise to that all the other fused transfer matrix  $t^{(j)}(u)$  can be expressed in terms of some sum of products of the fundamental one (see (3.3)). Particularly, the spin- $s$  transfer matrix  $t^{(s)}(u)$

(or the spin- $s'$  transfer matrix  $t^{(s')}(u)$ ) is expressed in terms of the sum of products of  $t^{(\frac{1}{2})}(u)$  with orders up to  $2s$  (or  $2s'$ ). The very identities (3.10)-(3.12) then lead to  $2N$  constraints on the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$ . Thus the relations (3.10)-(3.12) and (3.19)-(3.23) completely characterize the eigenvalues of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$  (as a consequence, also determine the eigenvalues of all the transfer matrices  $\{t^{(j)}(u)\}$ ).

## 3.2 Functional relations of the eigenvalues

The commutativity (2.22) of the fused transfer matrices  $\{t^{(j)}(u)\}$  with different spectral parameters implies that they have common eigenstates. Let  $|\Psi\rangle$  be a common eigenstate of these fused transfer matrices, which does not depend upon  $u$ , with the eigenvalue  $\Lambda^{(j)}(u)$ , i.e.,

$$t^{(j)}(u)|\Psi\rangle = \Lambda^{(j)}(u)|\Psi\rangle.$$

The fusion hierarchy relation (3.1) of the fused transfer matrices allows one to express all the eigenvalues  $\Lambda^{(j)}(u)$  in terms of the fundamental one  $\Lambda^{(\frac{1}{2})}(u)$  by the following recursive relations

$$\begin{aligned} \Lambda^{(\frac{1}{2})}(u) \Lambda^{(j-\frac{1}{2})}(u-j\eta) &= \Lambda^{(j)}(u - (j - \frac{1}{2})\eta) + \delta^{(s,s')}(u) \Lambda^{(j-1)}(u - (j + \frac{1}{2})\eta), \\ j &= \frac{1}{2}, 1, \frac{3}{2}, \dots \end{aligned} \quad (3.24)$$

Here  $\Lambda^{(0)}(u) = 1$  and the coefficient function  $\delta^{(s,s')}(u)$  is given by (3.2). The very operator identities (3.10) and (3.12) of the fused transfer matrix at the points  $\theta_j$  imply that the eigenvalue  $\Lambda^0(u)$  satisfies the similar relations

$$\Lambda^{(s)}(\theta_{2j-1}) \Lambda^{(s)}(\theta_{2j-1} - \eta) = \Delta^{(s)}(u) \big|_{u=\theta_{2j-1}}, \quad j = 1, \dots, N, \quad (3.25)$$

$$\Lambda^{(s')}(\theta_{2j}) \Lambda^{(s')}(\theta_{2j} - \eta) = \Delta^{(s')}(u) \big|_{u=\theta_{2j}}, \quad j = 1, \dots, N, \quad (3.26)$$

where the function  $\Delta^{(l)}(u)$  is given by (3.11). The properties of the transfer matrix  $t^{(\frac{1}{2})}(u)$  given by (3.19)-(3.23) give rise to that the corresponding eigenvalue  $\Lambda^{(\frac{1}{2})}(u)$  satisfies the following relations

$$\Lambda^{(\frac{1}{2})}(u + i\pi) = \Lambda^{(\frac{1}{2})}(u), \quad (3.27)$$

$$\Lambda^{(\frac{1}{2})}(-u - \eta) = \Lambda^{(\frac{1}{2})}(u), \quad (3.28)$$

$$\begin{aligned}
\Lambda^{(\frac{1}{2})}(0) &= -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \\
&\times \prod_{l=1}^N \sinh(\theta_{2l-1} + (\frac{1}{2} + s)\eta) \sinh(-\theta_{2l-1} + (\frac{1}{2} + s)\eta) \\
&\times \prod_{l=1}^N \sinh(\theta_{2l} + (\frac{1}{2} + s')\eta) \sinh(-\theta_{2l} + (\frac{1}{2} + s')\eta), \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
\Lambda^{(\frac{1}{2})}(\frac{i\pi}{2}) &= -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \\
&\times \prod_{l=1}^N \sinh(\frac{i\pi}{2} + \theta_{2l-1} + (\frac{1}{2} + s)\eta) \sinh(\frac{i\pi}{2} + \theta_{2l-1} - (\frac{1}{2} + s)\eta) \\
&\times \prod_{l=1}^N \sinh(\frac{i\pi}{2} + \theta_{2l} + (\frac{1}{2} + s')\eta) \sinh(\frac{i\pi}{2} + \theta_{2l} - (\frac{1}{2} + s')\eta), \tag{3.30}
\end{aligned}$$

$$\Lambda^{(\frac{1}{2})}(u) \big|_{u \rightarrow \pm\infty} = -\frac{\cosh(\theta_- - \theta_+) e^{\pm[(4N+4)u + (2N+2)\eta]}}{2^{4N+1}} + \dots \tag{3.31}$$

The analyticities of the spin- $(\frac{1}{2}, l)$   $R$ -matrix and spin- $\frac{1}{2}$   $K$ -matrices and the property (3.31) imply that the eigenvalue  $\Lambda^{(\frac{1}{2})}(u)$  possesses the following analytical property

$$\Lambda^{(\frac{1}{2})}(u), \text{ as a function of } u, \text{ is a trigonometric polynomial of degree } 4N + 4. \tag{3.32}$$

Namely,  $\Lambda^{(\frac{1}{2})}(u)$  is a trigonometric polynomial of  $u$  with  $4N + 5$  unknown coefficients. The crossing relation (3.28) reduces the number of the independent unknown coefficients to  $2N + 3$ . Therefore the relations (3.24)-(3.31) and the property (3.32) completely characterize the spectrum of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$ .

## 4 T-Q Ansatz and the associated BAEs

### 4.1 Generic boundary parameters

Following the method developed in [44], let us introduce the following inhomogeneous  $T - Q$  Ansatz for the eigenvalue  $\Lambda^{(\frac{1}{2})}(u)$  of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$  basing on the

conditions (3.24)-(3.32) <sup>3</sup>,

$$\begin{aligned}\Lambda^{(\frac{1}{2})}(u) &= a^{(s,s')}(u) \frac{Q(u-\eta)}{Q(u)} + d^{(s,s')}(u) \frac{Q(u+\eta)}{Q(u)} \\ &\quad + 2c \sinh 2u \sinh(2u+2\eta) \frac{F^{(s,s')}(u)}{Q(u)},\end{aligned}\tag{4.1}$$

where the functions  $a^{(s,s')}(u)$ ,  $d^{(s,s')}(u)$ ,  $F^{(s,s')}(u)$  and the constant  $c$  are given by

$$\begin{aligned}a^{(s,s')}(u) &= -2^2 \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh(u-\alpha_-) \cosh(u-\beta_-) \\ &\quad \times \sinh(u-\alpha_+) \cosh(u-\beta_+) \bar{A}^{(s,s')}(u),\end{aligned}\tag{4.2}$$

$$d^{(s,s')}(u) = a^{(s,s')}(-u-\eta),\tag{4.3}$$

$$\begin{aligned}\bar{A}^{(s,s')}(u) &= \prod_{j=1}^N \sinh(u-\theta_{2j-1} + (\frac{1}{2}+s)\eta) \sinh(u+\theta_{2j-1} + (\frac{1}{2}+s)\eta) \\ &\quad \times \prod_{j=1}^N \sinh(u-\theta_{2j} + (\frac{1}{2}+s')\eta) \sinh(u+\theta_{2j} + (\frac{1}{2}+s')\eta),\end{aligned}\tag{4.4}$$

$$\begin{aligned}F^{(s,s')}(u) &= \prod_{j=1}^N \prod_{k=0}^{2s} \sinh(u-\theta_{2j-1} + (\frac{1}{2}-s+k)\eta) \\ &\quad \times \sinh(u+\theta_{2j-1} + (\frac{1}{2}-s+k)\eta) \\ &\quad \times \prod_{j=1}^N \prod_{k=0}^{2s'} \sinh(u-\theta_{2j} + (\frac{1}{2}-s'+k)\eta) \\ &\quad \times \sinh(u+\theta_{2j} + (\frac{1}{2}-s'+k)\eta),\end{aligned}\tag{4.5}$$

$$c = \cosh(\alpha_- + \beta_- + \alpha_+ + \beta_+ + (1+2(s+s')N)\eta) - \cosh(\theta_- - \theta_+).\tag{4.6}$$

The function  $Q(u)$  is parameterized by  $2(s+s')N$  parameters  $\{\lambda_j | j = 1, \dots, 2(s+s')N\}$  as follow

$$Q(u) = \prod_{j=1}^{2(s+s')N} \sinh(u-\lambda_j) \sinh(u+\lambda_j+\eta) = Q(-u-\eta).\tag{4.7}$$

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<sup>3</sup>It was shown in [44] (see also [49]) that there actually exist a variety of apparent different  $T-Q$  relations. However, different forms of these  $T-Q$  relations only give different parameterizations of the eigenvalues but not different states, and each of them gives the complete set of the eigenvalues. For any spins  $s, s'$  and the total number of sites  $2N$ , one can always choose a minimal number  $2(s+s')N$  of the Bethe parameters to parameterize  $\Lambda^{(\frac{1}{2})}(u)$  like (4.1).

From the explicit expressions (4.1)-(4.7) of the Ansatz for  $\Lambda^{(\frac{1}{2})}(u)$ , one can easily check that the  $T - Q$  Ansatz (4.1) does satisfy the relations (3.24)-(3.32) as follows. The explicit expressions (4.2)-(4.5) of the functions implies that

$$\begin{aligned} a^{(s,s')}(\theta_{2j-1} - (\frac{1}{2} + s)\eta) &= 0, \quad d^{(s,s')}(\theta_{2j-1} + (s - \frac{1}{2})\eta) = 0, \\ F^{(s,s')}(\theta_{2j-1} + (s - \frac{1}{2} - k\eta)) &= 0, \quad \text{for } k = 0, 1, \dots, 2s, \quad j = 1, \dots, N. \end{aligned}$$

Keeping the above equations in mind and iterating the fusion hierarchy relations (3.24) from the fundamental one  $\Lambda^{\frac{1}{2}}(u)$  given in terms of the inhomogeneous  $T$ - $Q$  relation (4.1), we can evaluate  $\Lambda^{(s)}(u)$  at the points  $\theta_{2j-1}$  and  $\theta_{2j-1} - \eta$  as

$$\Lambda^{(s)}(\theta_{2j-1}) = \frac{Q(\theta_{2j-1} - (s + \frac{1}{2})\eta)}{Q(\theta_{2j-1} + (s - \frac{1}{2})\eta)} \prod_{k=0}^{2s-1} a^{(s,s')}(\theta_{2j-1} + (s - \frac{1}{2} - k)\eta), \quad (4.8)$$

$$\Lambda^{(s)}(\theta_{2j-1} - \eta) = \frac{Q(\theta_{2j-1} + (s - \frac{1}{2})\eta)}{Q(\theta_{2j-1} - (s + \frac{1}{2})\eta)} \prod_{k=0}^{2s-1} d^{(s,s')}(\theta_{2j-1} + (s - \frac{3}{2} - k)\eta). \quad (4.9)$$

The above equations yield

$$\begin{aligned} \Lambda^{(s)}(\theta_{2j-1})\Lambda^{(s)}(\theta_{2j-1} - \eta) &= \prod_{k=0}^{2s-1} a^{(s,s')}(\theta_{2j-1} - (s - \frac{1}{2})\eta + k\eta) \\ &\quad \times d^{(s,s')}(\theta_{2j-1} - (s - \frac{1}{2})\eta + k\eta - \eta) \\ &= \prod_{k=0}^{2s-1} \delta^{(s,s')}(\theta_{2j-1} - (s - \frac{1}{2})\eta + k\eta) \\ &= \Delta^{(s)}(\theta_{2j-1}), \quad j = 1, \dots, N. \end{aligned} \quad (4.10)$$

In deriving the second equality of the above equation, we have used the following identity

$$\delta^{(s,s')}(u) = a^{(s,s')}(u) d^{(s,s')}(u - \eta). \quad (4.11)$$

Similarly, we can show that the anstaz (4.1) for  $\Lambda^{(\frac{1}{2})}(u)$  also satisfies the relation (3.26). This means that the  $T - Q$  Ansatz (4.1) indeed satisfies the very functional identities (3.25)-(3.26). From the explicit expression (4.1) one may find that the  $T - Q$  Ansatz might have some apparent simple poles at the following points:

$$\lambda_j, \quad -\lambda_j - \eta, \quad j = 1, \dots, 2(s + s')N.$$

The regularity of the transfer matrix implies that the residues of the  $T - Q$  Ansatz (4.1) at these points have to vanish which gives rise to the associated BAEs

$$\begin{aligned} & a^{(s,s)}(\lambda_j)Q(\lambda_j - \eta) + d^{(s,s')}(\lambda_j)Q(\lambda_j + \eta) \\ & + 2c \sinh 2\lambda_j \sinh(2\lambda_j + 2\eta)F^{(s,s')}(\lambda_j) = 0, \\ & j = 1, \dots, 2(s + s')N. \end{aligned} \quad (4.12)$$

Finally we conclude that the  $T - Q$  Ansatz (4.1) indeed satisfies (3.24)-(3.32) provided that the  $2(s + s')N$  parameters  $\{\lambda_j | j = 1, \dots, 2(s + s')N\}$  satisfy the associated BAEs (4.12). Thus the  $\Lambda^{(\frac{1}{2})}(u)$  given by (4.1) becomes the eigenvalue of the transfer matrix  $t^{(\frac{1}{2})}(u)$  given by (2.21) with  $j = \frac{1}{2}$ . With the help of the recursive relation (3.24), we can obtain the inhomogeneous  $T - Q$  equations <sup>4</sup> for all the other  $\Lambda^{(j)}(u)$  from that of the fundamental one  $\Lambda^{(\frac{1}{2})}(u)$ .

## 4.2 Constrained boundary parameters

Following [32], let us introduce four parameters  $\{\epsilon_i | i = 0, 1, 2, 3\}$ , each of which takes the values  $\pm 1$ . These discrete parameters satisfy the relation

$$\epsilon_1 \epsilon_2 \epsilon_3 = 1. \quad (4.13)$$

Similarly as that in previous subsection we can show that the eigenvalue  $\Lambda^{(\frac{1}{2})}(u)$  of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$  can also be parameterized by any of the following inhomogeneous  $T - Q$  relations,

$$\begin{aligned} \Lambda^{(\frac{1}{2})}(u) &= a_{\pm}^{(s,s')}(u | \epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u - \eta)}{Q(u)} + d_{\pm}^{(s,s')}(u | \epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u + \eta)}{Q(u)} \\ &+ 2c_{\pm}(\epsilon_1, \epsilon_2, \epsilon_3) \sinh 2u \sinh(2u + 2\eta) \frac{F^{(s,s')}(u)}{Q(u)}, \end{aligned} \quad (4.14)$$

where the functions  $F^{(s,s')}(u)$  and  $Q(u)$  are given by (4.5) and (4.7), and the functions  $a_{\pm}^{(s,s')}(u)$  and  $d_{\pm}^{(s,s')}(u)$  are

$$a_{\pm}^{(s,s')}(u | \epsilon_1, \epsilon_2, \epsilon_3) = -2^2 \epsilon_2 \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \sinh(u \mp \alpha_-) \cosh(u \mp \epsilon_1 \beta_-)$$

---

<sup>4</sup>It is noted that for the spin- $\frac{1}{2}$  open chain, such equations can be obtained from a generating functions [74].



$$\times \sinh(u \mp \epsilon_2 \alpha_+) \cosh(u \mp \epsilon_3 \beta_+) \bar{A}^{(s,s')}(u), \quad (4.15)$$

$$d_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3) = a_{\pm}^{(s,s')}(-u - \eta|\epsilon_1, \epsilon_2, \epsilon_3). \quad (4.16)$$

The constant  $c_{\pm}(\epsilon_1, \epsilon_2, \epsilon_3)$  is

$$c_{\pm} = \epsilon_2 \cosh(\alpha_- + \epsilon_1 \beta_- + \epsilon_2 \alpha_+ + \epsilon_3 \beta_+ \pm (1 + 2(s + s')N)\eta) - \cosh(\theta_- - \theta_+). \quad (4.17)$$

The regularity of each  $T - Q$  Ansatz (4.14) at the roots of  $Q(u)$  gives rise to the resulting BAEs which are similar as those of (4.12). It is easy to check that each  $T - Q$  relation (4.14) indeed satisfies (3.24)-(3.32). The numerical analysis [74, 45] of the solutions of the associated BAEs for some small sites  $N$  with  $s = s'$  strongly suggests that each of the  $T - Q$  relation (4.14) may give a complete set of solutions of the transfer matrix.

In order to look for the solution of which the third term (i.e. inhomogeneous term) of (4.14) vanishes, namely, the solution of the form

$$\Lambda^{(\frac{1}{2})}(u) = a_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u - \eta)}{Q(u)} + d_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u + \eta)}{Q(u)}, \quad (4.18)$$

the asymptotic behavior (3.31) gives rise to that the 6 boundary parameters  $\alpha_{\pm}$ ,  $\beta_{\pm}$  and  $\theta_{\pm}$  have to satisfy the corresponding constraint

$$\alpha_- + \epsilon_1 \beta_- + \epsilon_2 \alpha_+ + \epsilon_3 \beta_+ + \epsilon_0(\theta_- - \theta_+) = k\eta + \frac{1 - \epsilon_2}{2}i\pi \pmod{2i\pi}, \quad (4.19)$$

where  $k$  is an integer such that

$$2(s + s')N - 1 \mp k = 2M^{\pm}, \quad M^{\pm} = 0, 1, \dots \quad (4.20)$$

It is easy to check that each  $T - Q$  relation (4.18) indeed satisfies (3.24)-(3.30) and (3.32). Moreover, the constraint (4.19)-(4.20) makes (3.31) satisfied. This means that if the 6 boundary parameters  $\alpha_{\pm}$ ,  $\beta_{\pm}$  and  $\theta_{\pm}$  satisfy any of the constraints (4.19)-(4.20) the inhomogeneous  $T - Q$  relation (4.14) reduces to two conventional  $T - Q$  relations <sup>5</sup>

$$\begin{aligned} \Lambda_{\pm}^{(\frac{1}{2})}(u) &= a_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(\pm)}(u - \eta)}{Q^{(\pm)}(u)} \\ &\quad + d_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(\pm)}(u + \eta)}{Q^{(\pm)}(u)}, \end{aligned} \quad (4.21)$$

---

<sup>5</sup>The solutions with the discrete parameters  $\{\epsilon_i | i = 0, 1, 2, 3\}$  was first given in [32] and then discussed in [52].

where the functions  $a_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3)$  and  $d_{\pm}^{(s,s')}(u|\epsilon_1, \epsilon_2, \epsilon_3)$  are given by (4.15) and (4.16) and the  $Q$ -functions  $Q^{(\pm)}(u)$  are respectively,

$$Q^{(\pm)}(u) = \prod_{j=1}^{M^{\pm}} \sinh(u - \lambda_j^{\pm}) \sinh(u + \lambda_j^{\pm} + \eta),$$

$$M^{\pm} = \frac{1}{2}(2(s + s')N - 1 \mp k). \quad (4.22)$$

The parameters  $\{\lambda_j^{\pm}\}$  have to satisfy the conventional BAEs

$$\frac{a_{\pm}^{(s,s')}(\lambda_j^{\mp}|\epsilon_1, \epsilon_2, \epsilon_3)}{d_{\pm}^{(s,s')}(\lambda_j^{\mp}|\epsilon_1, \epsilon_2, \epsilon_3)} = -\frac{Q^{(\pm)}(\lambda_j^{\pm} + \eta)}{Q^{(\pm)}(\lambda_j^{\pm} - \eta)}, \quad j = 1, \dots, M^{\pm}. \quad (4.23)$$

## 5 Conclusions

The XXZ alternating spin chain with the generic non-diagonal boundary terms specified by the most general non-diagonal  $K$ -matrices given by (2.16)-(2.20) is studied by the off-diagonal Bethe ansatz method. Based on the intrinsic properties (2.13)-(2.14) of the fused  $R$ -matrices and  $K$ -matrices, we obtain the closed operator identities (3.10) and (3.12) of the fundamental transfer matrix  $t^{(\frac{1}{2})}(u)$ . These identities, together with other properties (3.19)-(3.23), allow us to construct the off-diagonal (or inhomogeneous)  $T - Q$  equation (4.1) and the associated BAEs (4.12) accounting for the eigenvalues of the transfer matrix.

We remark that if the anisotropic parameter  $\eta$  takes the following discrete values

$$\eta = -\frac{\alpha_- + \beta_- + \alpha_+ + \beta_+ \pm (\theta_- - \theta_+) + 2i\pi m}{2(s + s')N + 1}, \quad m \in \mathbb{Z}, \quad (5.1)$$

our inhomogeneous  $T$ - $Q$  relation (4.1) can be reduced to the conventional one

$$\Lambda^{(\frac{1}{2})}(u) = a^{(s,s')}(u) \frac{Q(u - \eta)}{Q(u)} + d^{(s,s')}(u) \frac{Q(u + \eta)}{Q(u)}, \quad (5.2)$$

where the  $Q$ -function is given by (4.7). The associated BAEs thus read

$$\frac{a^{(s,s')}(\lambda_j)}{d^{(s,s')}(\lambda_j)} = -\frac{Q(\lambda_j + \eta)}{Q(\lambda_j - \eta)}, \quad j = 1, \dots, 2(s + s')N. \quad (5.3)$$

When taking the thermodynamical limit  $N \rightarrow \infty$ <sup>6</sup>,  $\eta$  becomes dense on the imaginary line. This allows one to use the method developed in [46] to study the thermodynamic properties (up to the order of  $O(N^{-2})$ ) of the model for generic values of  $\eta$  via the conventional thermodynamic Bethe Ansatz methods [9].

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<sup>6</sup>At the same time, we also take the limit  $m \rightarrow \infty$  such that  $\eta$  be finite.

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## Appendix A: $U_q(sl_2)$ algebra and spin- $l$ representation

The underlying algebra of the XXZ spin chain is the quantum algebra  $U_q(sl_2)$  generated by  $\{S^\pm, S^3\}$  with the relations [8, 14, 15]

$$[S^+, S^-] = \frac{\sinh(2\eta S^3)}{\sinh \eta}, \quad [S^3, S^\pm] = \pm S^\pm. \quad (\text{A.1})$$

The Casimir operator  $C_2$  of  $U_q(sl_2)$ , which commutes with all the generators, is

$$\begin{aligned} C_2 &= \cosh(\eta + 2\eta S^3) + 2 \sinh^2 \eta S^- S^+ \\ &= \cosh(\eta - 2\eta S^3) + 2 \sinh^2 \eta S^+ S^-. \end{aligned} \quad (\text{A.2})$$

All the generators of  $U_q(sl_2)$  can be realized by  $(2l+1) \times (2l+1)$  matrices on  $(2l+1)$ -dimensional spin- $l$  space as follows

$$S^3|m\rangle = m|m\rangle, \quad (\text{A.3})$$

$$S^+|m\rangle = \sqrt{[l+1+m]_q[l-m]_q}|m+1\rangle, \quad (\text{A.4})$$

$$S^-|m\rangle = \sqrt{[l+m]_q[l+1-m]_q}|m-1\rangle, \quad m = l, l-1, \dots, -l, \quad (\text{A.5})$$

where  $\{|m\rangle|m = l, l-1, \dots, -l\}$  is an orthonormal basis of the spin- $l$  space and we have used the notation

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q = e^\eta.$$

The Casimir operator (A.2) acting on the spin- $l$  space is proportional to the identity operator, i.e.,

$$C_2|m\rangle = \cosh(\eta + 2l\eta)|m\rangle, \quad m = l, l-1, \dots, -l. \quad (\text{A.6})$$

For the simplest case, i.e.,  $l = \frac{1}{2}$ , the corresponding generators can be realized by  $2 \times 2$  matrices

$$S^\pm = \sigma^\pm, \quad S^3 = \frac{1}{2}\sigma^z, \quad C_2 = \cosh 2\eta \times \text{id}.$$

Moreover  $U_q(sl_2)$  is a Hopf algebra with the following coproduct  $\Delta$  [8]

$$\Delta(S^3) = S^3 \otimes \text{id} + \text{id} \otimes S^3, \quad (\text{A.7})$$

$$\Delta(S^\pm) = S^\pm \otimes q^{-S^3} + q^{S^3} \otimes S^\pm, \quad (\text{A.8})$$

which is an algebraic homomorphism and allows one to construct the representation on the tensor product of two representation spaces.

## Appendix B: Proofs of (2.13) and (2.14)

We restrict to the spin- $l$  space  $\mathbf{V}$  (i.e., a  $(2l+1)$ -dimensional vector space endowing a representation of  $U_q(sl_2)$  with the realization given by (A.3)-(A.5)). The  $R$ -matrix  $\bar{R}^{(l,l)}(u) \in \text{End}(\mathbf{V} \otimes \mathbf{V})$  of  $U_q(sl_2)$  with the coproduct given by (A.7)-(A.8) was given in [19] (for the rational case [68]),

$$\check{\bar{R}}^{(l,l)}(u) \equiv \mathbf{P} \bar{R}^{(l,l)}(u) = \prod_{k=1}^{2l} \sinh(-u + k\eta) \sum_{m=0}^{2l} \prod_{k=1}^m \frac{\sinh(u + k\eta)}{\sinh(-u + k\eta)} \bar{\mathbf{P}}^{(m)}, \quad (\text{B.1})$$

where  $\mathbf{P}$  is the permutation operator between the tensor product space of the spin- $l$  vector spaces,  $\bar{\mathbf{P}}^{(m)}$  is a projector acting on the tensor product space of two spin- $l$  spaces by the coproduct (A.7)-(A.8) and projects the tensor space into the irreducible subspace of spin- $m$  (i.e.,  $(2m+1)$ -dimensional subspace). In particular, the  $R$ -matrix  $\bar{R}^{(l,l)}(u)$  also satisfies the following important properties

$$\text{Initial condition : } \bar{R}_{12}^{(l,l)}(0) = \prod_{k=1}^{2l} \sinh(k\eta) \mathbf{P}_{12}, \quad (\text{B.2})$$

$$\text{Fusion condition : } \bar{R}_{12}^{(l,l)}(-\eta) = \prod_{k=2}^{2l+1} \sinh(k\eta) \mathbf{P}_{12} \bar{\mathbf{P}}_{12}^{(0)}(l). \quad (\text{B.3})$$

The projector  $\bar{\mathbf{P}}^{(0)}(l)$  projects the tensor space of two spin- $l$  spaces to the singlet state, namely,

$$\bar{\mathbf{P}}^{(0)}(l) = \frac{|\bar{\Phi}_0^{(l)}\rangle\langle\bar{\Phi}_0^{(l)}|}{\langle\bar{\Phi}_0^{(l)}|\bar{\Phi}_0^{(l)}\rangle}, \quad |\bar{\Phi}_0^{(l)}\rangle = \sum_{k=0}^{2l} (-q)^k |l-k\rangle \otimes |-l+k\rangle. \quad (\text{B.4})$$

The singlet state  $|\bar{\Phi}_0^{(l)}\rangle$  of  $U_q(sl_2)$  is the unique vector (up to a rescaling) which enjoys the following properties

$$\Delta(S^3)|\bar{\Phi}_0^{(l)}\rangle = \Delta(S^\pm)|\bar{\Phi}_0^{(l)}\rangle = 0. \quad (\text{B.5})$$

It was shown in [71] that the  $R$ -matrices  $R^{(l,l)}(u)$  and  $\bar{R}^{(l,l)}(u)$  have the following relation

$$R^{(l,l)}(u) = \left(\text{id} \otimes e^{uS^3}\right) \bar{R}^{(l,l)}(u) \left(\text{id} \otimes e^{-uS^3}\right). \quad (\text{B.6})$$

The above relation and the properties (B.2)-(B.4) imply that the fused spin- $(l, l)$   $R$ -matrix  $R^{(l,l)}(u)$  given by (2.11) with  $j = l$  indeed satisfies the very relations (2.13) and (2.14).

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